

BUMPY METRIC THEOREM IN THE SENSE OF MAÑÉ FOR NON-CONVEX HAMILTONIANS

SHAHRIAR ASLANI AND PATRICK BERNARD

ABSTRACT. We prove a bumpy metric theorem in the sense of Mañé for non-convex Hamiltonians that are satisfying a certain geometric property.

1. INTRODUCTION

In the study of fibewise convex Hamiltonian systems, Ricardo Mañé introduced the notion now called Mañé genericity ([13]): A property is called Mañé generic if, for each Hamiltonian H , the property is satisfied by the Hamiltonian $H + u$ for a generic potential u .

Although this notion is particularly relevant in the case where H is convex, it also makes perfect sense for more general Hamiltonians. Our goal in the present paper is to investigate the Mañé-generic properties of periodic orbits of not necessarily convex Hamiltonian systems and to work in the direction of what could be called a bumpy metric theorem in this context: the property that all periodic orbits on a given energy surface are non-degenerate is Mañé generic. In the convex case, this problem was studied in dimension 2 by Oliveira in [14], and the missing perturbation lemma necessary to generalize his result in any dimension was obtained in [15] (using an incorrect normal form fixed in [5]). Observe however that there is a case not treated in [14, 15], and that the bumpy metric theorem in the sense of Mañé remains open even in the convex case, more details below. In the present paper, we generalize the results proved in [14, 15] (which, as we just explained, are not exactly those stated) to non-convex Hamiltonians.

There is a long history on the study of generic properties of periodic orbits in various contexts. It started with the works of Kupka [10] and Smale [18] stating that all orbits are hyperbolic for generic systems in the class of all vectorfields (and moreover the intersections of stable and unstable manifolds are transverse). In the class of Hamiltonian systems, such a property can't be expected, but Robinson [16, 17] proved among other things that generic Hamiltonian systems have no degenerate periodic orbits on a given energy surface. The difference in the present work is that we are considering the restricted class of perturbations by potentials. The bumpy metric theorem, obtained in the same period, claims that generic geodesic flows have no degenerate periodic orbits, see [2, 11, 4] for contributions to this result, and [6] for the semi-Riemannian case. Perturbing by potentials is similar (and to a large extent equivalent) to considering conformal perturbations of metrics, which is a much smaller family of perturbations than the class of all metrics. As to the study of Mañé perturbations of convex Hamiltonians, besides the works [14, 15, 5] already discussed, let us mention Contreras [7] where some important tools are introduced.

We will say that the Hamiltonian $H(q, p) : T^*M \rightarrow \mathbb{R}$ is convex if its fiberwise Hessian $\partial_{p^2}^2 H(q, p)$ is positive-definite for all $(q, p) \in T^*\mathbb{R}^n$. The following definition will play a central role in our study of Mañé generic properties of periodic orbits :

Definition 1. *Let $H(q, p) : T^*M \rightarrow \mathbb{R}$ be a Hamiltonian which is defined on the cotangent bundle of a smooth manifold M . We say H is fiberwise iso-energetically non-degenerate at $(q, p) \in T^*M$ if*

$$(1.1) \quad \det \begin{bmatrix} \partial_{p^2}^2 H(q, p) & \partial_p H(q, p) \\ \partial_p H(q, p)^T & 0 \end{bmatrix} \neq 0.$$

This is equivalent to saying that p is a regular point of the function $H(q, \cdot)$ on T_q^*M , and that the Hessian of this function is non-degenerate on the kernel of its differential. This kernel is the intersection between the tangent space of the energy level and the fibre.

A convex Hamiltonian is fiberwise iso-energetically non-degenerate at each point except those where $\partial_p H = 0$. There is one such point per fiber.

Another important example to have in mind is the case of fiberwise quadratic Hamiltonians. If such a Hamiltonian is non degenerate in each fiber, then it is fiberwise isoenergetically non-degenerate precisely outside of its zero energy level.

Given a periodic orbit θ of the Hamiltonian H , we can as is usual take a transverse section and consider the Poincaré return map to that section. This Poincaré map preserves the energy level, and we call restricted Poincaré map its restriction to the energy level. The differential at the orbit of this restricted map is called the restricted linearized return map. It is well defined and symplectic on the tangent space to the restricted section. By taking a symplectic base of this tangent space, we can consider the restricted linearized return map as an element of the symplectic group $Sp(2d)$. Up to conjugacy, it does not depend on the section or the base. The orbit is called non-degenerate if 1 is not an eigenvalue of the restricted linearized map.

Given a periodic orbit $\theta(t) = (Q(t), P(t))$, of minimal period T , we say that s is a neat time if $\dot{Q}(s) \neq 0$ and if there exists no $t \neq s \pmod T$ such that $Q(s) = Q(t)$. With this definition, the set of neat times is easily seen to be open (see the proof of Lemma 6 below). Note, in contrast to what is implicitly assumed in [14, 15], that periodic orbits without any neat time may exist, even in the convex case. For example, if H is a natural system of the form $H(q, p) = g_q(p, p) + u(q)$, where g is a Riemannian metric and u is a potential defined on the base, then there often exist reversible periodic orbits, *i.e.* periodic orbits which perform a round trip above an arc in the base. The existence of such orbits is studied for example in [11], where they are called librations. They have no neat time.

Given an orbit θ of the Hamiltonian H , we say that the potential u is admissible for θ if the value and the differential of u are both zero at each point of the projection of θ . We denote by $C_\theta^\infty(M) \subset C^\infty(M)$ the space of admissible potentials. If u is admissible, then θ is also a periodic orbit of $H + u$. Given a transverse section to θ , the restricted transverse section of θ for $H + u$ is different from the restricted transverse section of θ for H , but they have the same tangent space at the orbit. So by choosing a symplectic base of this fixed tangent space, we can define the restricted linearized return map $L(\theta, H + u)$, $u \in C_\theta^\infty(M)$ as an element of $Sp(2d)$. The map $u \mapsto L(\theta, H + u)$ is well defined up to a fixed conjugacy. Our first result is:

Theorem 1. *Consider a smooth Hamiltonian $H(q, p) : T^*M \rightarrow \mathbb{R}$, and a periodic orbit $\theta(t)$ of the Hamiltonian vector field of H . Assume that θ admits a neat time $t_0 \in \mathbb{R}$ such that H is fiberwise iso-energetically non-degenerate at $\theta(t_0)$. Then the map*

$$C_\theta^\infty(M) \ni u \mapsto L(\theta, H + u) \in Sp(2d)$$

is weakly open, meaning that the image of each non-empty open set contains a non-empty open set.

This theorem is proved below in Section 3 using a local normal form stated and proved in Section 2. It is likely that the above map is actually open, an adaptation [12] should provide a proof. In the convex case, this perturbation statement was obtained in [15], using a normal form from [8] which was corrected in [5]. Our proof follows a similar strategy. In the convex case, it is automatic that H is fiberwise isoenergetically non-degenerate at $\theta(t_0)$ when t_0 is a neat time, so this assumption can be omitted. However, the assumption that θ admits a neat time is necessary, and not automatic. It is wrongly omitted in [14, 15]. To see that the existence of a neat time is a necessary assumption, consider again a natural system $H = g_q(p, p) + v(q)$ and an orbit θ which is a libration (a reversible orbit). Then $H + u$ is still reversible for each potential u , meaning that $(H + u)(q, -p) = (H + u)(q, p)$ and as a consequence $L(\theta, H + u)$ is a reversible symplectic matrix for each $u \in C_\theta^\infty(M)$, meaning that $R = LRL$, where $R = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Since the space of symplectic reversible matrices is a submanifold of positive codimension in $Sp(2d)$, the image of the map $u \mapsto L(\theta, H + u)$ has no interior, which contradicts the conclusion of the theorem in that case.

It is also clear that some nondegeneracy assumption on H is necessary. Indeed, we can consider the Hamiltonian $H = p_1$ on the manifold $M = \mathbb{T} \times \mathbb{R}^d$ with coordinates (q_1, \hat{q}) . All the Hamiltonians $H + u$ generate the equations $q_1' = 1, \hat{q}' = 0$, so all orbits are periodic, and, in the sections q_1 constant, the return map is fixing the \hat{q} coordinate, hence the linearized return maps have a first block line equal to $[I, 0]$. Such matrices have no interior in $Sp(2d)$.

By methods similar to those used in [1, 2, 4, 14], Theorem 1 implies :

Theorem 2. *Given a smooth Hamiltonian $H : T^*M \rightarrow \mathbb{R}$, and a Baire meager subset $\Upsilon \subset Sp(2d)$ invariant under conjugacy, there exists a dense G_δ subset $\mathcal{G} \subset C^\infty(M)$ such that, for each $u \in \mathcal{G}$, the Hamiltonian system $H + u$ has the following property :*

The zero energy level of $H + u$ is regular. Moreover, if θ is a zero energy periodic orbit of $H + u$ which admits a neat time t_0 such that H is fiberwise isoenergetically non-degenerate at $\theta(t_0)$, then θ is non-degenerate and satisfies

$$L(\theta, H + u) \notin \Upsilon.$$

This theorem is proved in section 4. The main lines in the proof are similar to those used in [4, 14], but our presentation is different and avoids the recurrence on the periods.

The conclusion that θ is non-degenerate is actually contained in the conclusion $L(\theta, H + u) \notin \Upsilon$, provided Υ contains matrices having an eigenvalue equal to one, which can be assumed without loss of generality.

As mentioned earlier, the convex case was obtained in dimension 2 ($d = 1$) by Oliveira in [14], and then in any dimension by Rifford and Ruggiero in [15]

(complemented by [5]). Our statement is weaker than those in these papers, where the conclusions are claimed for all periodic orbits. However, the existence of a neat time t_0 is implicitly used in these papers, and it is wrongly omitted in the statements. In the convex case, once t_0 is a neat time, it is automatic that H is fiberwise isoenergetically non-degenerate at $\theta(t_0)$.

Relaxing the constraint that θ admits a neat time and obtaining a result for all periodic orbits is thus an interesting problem which remains open even in the convex case, it will not be discussed further in the present paper .

In the non-convex situation studied here, we have the second undesirable constraint that H be fiberwise isoenergetically non-degenerate at $\theta(t_0)$. This can obviously be ensured by assuming that H is fiberwise isoenergetically non-degenerate at each point x of T^*M where $\partial_p H(x) \neq 0$. This however is a strong assumption which is satisfied in the convex case, but not when H is a non-convex fiberwise quadratic Hamiltonian. An important observation here is that the set Σ_H of points of T^*M at which H fails to be fiberwise isoenergetically non-degenerate is a fixed data of the problem, which is unchanged by adding a potential. We now give a hypothesis on this set Σ_H which is sufficient to generically ensure the condition that H is fiberwise isoenergetically non-degenerate at $\theta(t_0)$. This condition is satisfied by the set Σ_H associated to non-degenerate fiberwise quadratic Hamiltonians.

Hypothesis 1. *The subset $\Sigma \subset T^*M$ is contained in a countable union of manifolds of positive codimension which are transversal to the vertical.*

Theorem 3. *Let $H : T^*M \rightarrow \mathbb{R}$ be a smooth Hamiltonian and let Σ be a subset of T^*M satisfying Hypothesis 1. There exists a dense G_δ subset $\mathcal{G} \subset C^\infty(M)$ such that, for each $u \in \mathcal{G}$, the Hamiltonian system $H + u$ has the following property :*

For each orbit θ of $H + u$, and each time t_0 such that $\partial_p H(\theta(t_0)) \neq 0$, there exist times t , arbitrarily close to t_0 , such that $\theta(t) \notin \Sigma$.

This theorem is proved in Section 5. It can be applied in particular to the case where Σ is the subset of points at which H is not fiberwise isoenergetically non-degenerate, and we obtain:

Theorem 4. *Let $H : T^*M \rightarrow \mathbb{R}$, be a Hamiltonian such that Σ_H satisfies Hypothesis 1, and let $\Upsilon \subset Sp(2d)$ be Baire meager subset invariant under conjugacy. There exists a dense G_δ subset $\mathcal{G} \subset C^\infty(M)$ such that, for each $u \in \mathcal{G}$, the Hamiltonian system $H + u$ has the following property :*

The zero energy level of $H + u$ is regular. Moreover, if θ is a zero energy periodic orbit of $H + u$ which admits a neat time t_0 , then θ is non-degenerate and satisfies

$$L(\theta, H + u) \notin \Upsilon.$$

Notations. We denote by $d+1$ the dimension of M , so that the symplectic sections have dimension $2d$. We denote by $\mathcal{M}(m)$, $\mathcal{S}(m)$, $\mathcal{S}^-(m)$ respectively the spaces of square matrices of size m , of symmetric matrices of size m , and of antisymmetric matrices of size m . Finally, we denote by $Sp(m)$ (m even) the group of symplectic matrices and by $\mathfrak{sp}(m)$ the space of Hamiltonian matrices, which are the matrices

L such that $\mathbb{J}L$ is symmetric, where \mathbb{J} is the standard symplectic matrix $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

We will usually denote by $q = (q_1, \dots, q_{d+1}) = (q_1, \hat{q})$ the local coordinates on M and by $x = (q, p) = (x_1, \hat{x})$ the local coordinates on T^*M . We denote by (e_i) the canonical bases of \mathbb{R}^m and \mathbb{R}^{m*} .

We will denote by $\varphi(t, x, u)$ or $\varphi_u^t(x)$ the Hamiltonian flow of $H + u$. Note that the space $C^\infty(M)$ of potentials u is a separable Fréchet space, but not a Banach space. Although names may suggest the opposite, there is not simple notion of Fréchet differential on a Fréchet space. The notation ∂_u of partial derivative with respect to the variable $u \in C^\infty(M)$ will always be understood in the meaning of Gateau differentiability. We will restrict to finite dimensional subspaces of $C^\infty(M)$ whenever we really manipulate differential calculus (and then refer to the stronger notion of Fréchet differential).

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2. NORMAL FORM NEAR ORBIT SEGMENTS OF NON-CONVEX HAMILTONIAN SYSTEMS

In this section, we give a normal form near fiberwise isoenergetically non-degenerate points which will be used to derive Theorem 1. This normal form and its proof are similar to the one obtained in [5] in the convex setting, and the novelty here is to single out fiberwise isoenergetic non-degeneracy as the appropriate hypothesis. Since potentials, or in other words Hamiltonians constant on the fibers, play a special role in the problem, we need to restrict to symplectic charts which preserve the vertical fibration, and more precisely charts $\Psi : U \times (\mathbb{R}^{d+1})^* \rightarrow T^*M$ which send each fiber $q \times (\mathbb{R}^{d+1})^*$ to a fiber of T^*M (not necessarily in a linear way).

Theorem 5. *Let $H : T^*M \rightarrow \mathbb{R}$ be a smooth Hamiltonian, and let $x_0 \in T^*M$ be a point such that H is fiberwise isoenergetically non-degenerate at x_0 . Then, there exists a fibered symplectic chart near x_0 and $\delta > 0$ such that, in these coordinates, for $|t| \leq \delta$, we have:*

- (1) $\varphi_H^t(0) = (te_1, 0)$
- (2) $\partial_{q\bar{p}}^2 H(te_1, 0) = 0.$
- (3) $\partial_{p_1\bar{p}}^2 H(te_1, 0) = 0.$
- (4) $\partial_{\bar{p}\bar{p}}^2 H(te_1, 0) = D,$

where D is a diagonal matrix whose diagonal elements are ± 1 , and where we denote by φ_H^t the Hamiltonian flow of H .

Remark 1. Assertion (1) implies that

$$(5) \quad \partial_{q_1 q}^2 H(te_1, 0_n) = 0, \quad t \in [-\delta, \delta].$$

Proof of theorem 5. By using a chart on M at $q_0 = \pi(x_0)$, we can assume that $M = \mathbb{R}^{d+1}$. Since H is fiberwise isoenergetically non-degenerate at x_0 , we have $\partial_p H(x_0) \neq 0$, which implies that the projected orbit $t \mapsto \pi \circ \varphi_H^t(x_0)$ is an embedding near $t = 0$. This allows to take the chart in such a way that, for small t ,

$$(2.1) \quad \pi \circ \varphi_H^t(x_0) = te_1.$$

We denote by $\underline{H}(q, p)$ the Hamiltonian in these local coordinates. Since $\partial_p \underline{H} = e_1$ the hypothesis that H is fiberwise isoenergetically non-degenerate at x_0 is equivalent in coordinates to the fact that $\partial_{\bar{p}\bar{p}}^2 \underline{H}(0)$ is invertible.

Definition 2. (a) We call a symplectic map $\Psi(q, p) : T^*\mathbb{R}^{d+1} \rightarrow T^*\mathbb{R}^{d+1}$ fibered whenever it preserve the vertical fibration. The symplectic diffeomorphism Ψ is fibered if and only if it has the form $\Psi(q, p) = (\phi(q), G(q, p))$, where $\phi(q)$ is a local diffeomorphism of the \mathbb{R}^{d+1} .

(b) We say that Ψ is homogeneous if it preserves the zero section, then its is of the form $\Psi(q, p) = (\phi(q), [d\phi^{-1}(q)]^T p)$ for some local diffeomorphism ϕ of the base.

(c) We say that Ψ is vertical if it preserves each fiber, then it is of the form $\Psi(q, p) = (q, p + dg(q))$ for some function $g(q) : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$.

(d) We say that Ψ is admissible if it is a fibered symplectic local diffeomorphism, and if its horizontal component is the identity on a small interval $te_1, |t| \leq \delta$.

Each fibered symplectic diffeomorphism is the composition of a homogeneous and of a vertical one.

We shall now prove the conclusions of the Theorem by applying a succession of admissible diffeomorphisms. In each step, we denote by \underline{H} the original Hamiltonian, and by $H = \underline{H} \circ \Psi$ the Hamiltonian in the new coordinates. We allow ourselves to reduce δ at each step.

Proof of (1). Let $\underline{P}_1(t)$ be the first component of $\underline{P}(t)$ (the vertical component of the orbit before the change of coordinates). We consider a function $v(t) : \mathbb{R} \rightarrow \mathbb{R}$ such that $v' = \underline{P}_1$ and the function $u(q_1, \hat{q}) := v(q_1) + \hat{P}(q_1) \cdot \hat{q}$. We have $du_{te_1} = \underline{P}(t)$, hence applying the vertical diffeomorphism $\Psi(q, p) = (q, p + du_q)$, the new orbit $(Q(t), P(t)) = \Psi^{-1}(\underline{Q}(t), \underline{P}(t))$ satisfies $P(t) = 0$ for small t . \square

Proof of (2). We assume that (1) holds. We consider the vector field

$$\underline{V}(q) := \partial_p \underline{H}(q, 0)$$

on \mathbb{R}^{d+1} . We apply the flow box Theorem to find a local diffeomorphism ϕ of \mathbb{R}^{d+1} near 0 such that the backward image of \underline{V} by ϕ is the constant vectorfield $V(q) \equiv e_1$. Since the initial orbit of 0 for the flow of \underline{V} is already $t \mapsto te_1$, we can moreover assume that ϕ is fixing the points $te_1, |t| \leq \delta$.

Let $\Psi(q, p) = (\phi(q), p \circ d\phi_q^{-1})$ be the corresponding homogeneous diffeomorphism and $H = \underline{H} \circ \psi$. Since Ψ is symplectic, it sends the Hamiltonian vectorfield X_H of H to the Hamiltonian vectorfield $X_{\underline{H}}$ of \underline{H} , meaning that

$$X_{\underline{H}}(\Psi(q, p)) = d\Psi_{(q,p)} \cdot X_H(q, p),$$

and in particular

$$X_{\underline{H}}(\phi(q), 0) = d\Psi_{(q,0)} \cdot X_H(q, 0).$$

Looking at the horizontal components, we get

$$\underline{V}(\phi(q)) = d\phi_q \cdot \partial_p H(q, 0)$$

which means that $V(q) := \partial_p H(q, 0)$ is the backward image of \underline{V} under ϕ . The way we chose ϕ implies that

$$\partial_p H(q, 0) = V(q) \equiv e_1$$

locally. This is much stronger than (2), but we will apply other changes of coordinates which will not preserve this additional structure. \square

The two steps just performed do not use the hypothesis of fiberwise isoe-nergetic non-degeneracy, but this hypothesis is essential for the next steps. In coordinates such that (1) holds, this hypothesis is equivalent to the invertibility of the matrix $\partial_{pp}^2 H(0)$.

Proof of (3). We assume that (1) and (2) are satisfied for \underline{H} , and prove that (3) can be obtained by a further admissible change of coordinates. We consider a base diffeomorphism of the form $\phi(q_1, \hat{q}) = (q_1 + l(q_1) \cdot \hat{q}, \hat{q})$, where $q_1 \mapsto l(q_1)$ is a smooth map with values in \mathbb{R}^{d*} defined near 0 in \mathbb{R} . The corresponding homogeneous diffeomorphism satisfies

$$\Psi : (q_1, 0, p_1, \hat{p}) \mapsto (q_1, 0, p_1, \hat{p} - p_1 l(q_1)).$$

We then have $\partial_{\hat{p}}(\underline{H} \circ \Psi)_{(q_1 e_1, p_1, \hat{p})} = \partial_{\hat{p}} \underline{H}_{(q_1 e_1, p_1, \hat{p} - p_1 l(q_1))}$ hence

$$\partial_{p_1 \hat{p}}^2 H_{(q_1 e_1, 0)} = \partial_{p_1 \hat{p}}^2 (\underline{H} \circ \Psi)_{(q_1 e_1, 0)} = \partial_{p_1 \hat{p}}^2 \underline{H}_{(q_1 e_1, 0)} - \partial_{\hat{p} \hat{p}}^2 \underline{H}_{(q_1 e_1, 0)} \cdot l(q_1).$$

We obtain (3) by choosing

$$l(q_0) := (\partial_{\hat{p} \hat{p}}^2 \underline{H}_{(q_1 e_1, 0)})^{-1} \cdot \partial_{p_1 \hat{p}}^2 \underline{H}_{(q_1 e_1, 0)}.$$

Observe that $\partial_{\hat{p} \hat{p}}^2 \underline{H}_{(q_1 e_1, 0)}$ is invertible for small q_1 as a consequence of the hypothesis of fiberwise isoenergetic non-degeneracy.

Finally, let us check that we have preserved (2). As in the proof of (2), the diffeomorphism ϕ sends the vectorfield $V(q) = \partial_p H(q, 0)$ to the vectorfield $\underline{V}(q) = \partial_p \underline{H}(q, 0)$. Here ϕ preserves the \hat{q} coordinate, hence

$$\partial_{\hat{q} p}^2 H(q_1 e_1, 0) = \partial_{\hat{q}} V(q_1 e_1) = \partial_{\hat{q}} \underline{V}(q_1 e_1) = \partial_{\hat{q} p}^2 \underline{H}(q_1 e_1, 0).$$

Note however that the additional equality $\partial_{q_1 p}^2 \underline{H}(q_1 e_1, 0) = 0$ that had been obtained in the previous step is not preserved. \square

Proof of (4). We assume that the equations (1) to (3) initially hold. We will obtain (4) by an admissible (usually not homogeneous) transformation preserving all these equalities. This transformation will be decomposed into first a homogeneous transformation and second a vertical transformation none of which preserve (2).

The first step consists of applying the homogeneous change of coordinates Ψ associated to a diffeomorphism of the form

$$\phi(q_1, \hat{q}) = (q_1, M(q_1) \cdot \hat{q}),$$

where $M(t)$ is a $d \times d$ invertible matrix depending smoothly on $t \in \mathbb{R}$ near $t = 0$. The matrix of the differential of ϕ is

$$J(q) = \begin{bmatrix} 1 & 0 \\ M'(q_1) \hat{q} & M(q_1) \end{bmatrix}, \quad J^{-1}(q) = \begin{bmatrix} 1 & 0 \\ -M^{-1}(q_1) M'(q_1) \hat{q} & M^{-1}(q_1) \end{bmatrix},$$

where $M'(q_1)$ is the derivative. We thus have

$$\Psi(q, p) = (q_1, M(q_1) \hat{q}, p_1 - \hat{p} M^{-1}(q_1) M'(q_1) \hat{q}, \hat{p} M^{-1}(q_1)).$$

The Hamiltonian in original coordinates is of the form

$$\underline{H}(q, p) = \underline{H}(q, 0) + \underline{v}(q) p_0 + \frac{1}{2} \underline{a}(q_1) p_1^2 + \frac{1}{2} \langle \hat{p} \underline{A}(q_1), \hat{p} \rangle + O_3(\hat{q}, p),$$

with $\underline{v}(q) = \partial_{p_1} \underline{H}(q, 0)$, $\underline{a}(q_1) = \partial_{p_1 p_1}^2 \underline{H}(q_1 e_1, 0)$, $\underline{A}(q_1) = \partial_{\hat{p} \hat{p}}^2 \underline{H}(q_1 e_1, 0)$. We compute

$$\begin{aligned} H(q, p) &= \underline{H} \circ \Psi(q, p) = H(q, 0) + v(q) p_1 - p_* M^{-1}(q_1) M'(q_1) \hat{q} + \frac{1}{2} a(q_1) p_1^2 \\ &\quad + \frac{1}{2} \langle \hat{p} M^{-1}(q_1) \underline{A}(q_1), \hat{p} M^{-1}(q_1) \rangle + O_3(\hat{q}, p). \end{aligned}$$

Since $\underline{A}(q_1)$ is assumed invertible for $q_1 = 0$, there exists a diagonal matrix D with diagonal terms equal to ± 1 , and a matrix $M(0)$ such that $M(0) D M^t(0) = \underline{A}(0)$,

which implies that $\partial_{\hat{p}\hat{p}}^2 H(0,0) = D$. Moreover, there exists a smooth curves $M(t)$ of matrices, defined near $t = 0$ such that $M(t)DM^t(t) = \underline{A}(t)$ for all small t (we shall actually construct such a curve $M(t)$ below) and this implies that $A(t) := \partial_{\hat{p}\hat{p}}^2 H(te_1, 0)$ is constant and equal to D for small t .

However, the unavoidable apparition of the term $\hat{p}M^{-1}(q_0)M'(q_0)\hat{q}$ means that (2) has been destroyed. In order to be able to restore it by a vertical change of coordinates, we need a particular choice for the curve $M(t)$:

Lemma 2. *We can choose $M(t)$ in such a way that*

$$H(q, p) = H(q, 1) + v(q)p_1 - \hat{p}DB(q_1)\hat{q} + \frac{1}{2}a(q_1)p_1^2 + \frac{1}{2}\hat{p}D\hat{p}^t + O_3(\hat{q}, p),$$

where $B(q_1)$ is symmetric for all small q_1 .

Proof. We need the matrix $M(t)$ to satisfy the two conditions that $M(t)DM^t(t) = \underline{A}(t)$ and $B(t) := DM^{-1}(t)M'(t)$ is symmetric (recall that $D^2 = Id$). Derivating the first condition, we get $M'DM^t + MD(M')^t = \underline{A}'$. Using the symmetry of B , we have that $(M^t)'(M^t)^{-1}D = DM^{-1}M'$, which implies $MD(M^t)' = M'DM^t$. We obtain the equation $2M'DM^t = \underline{A}'$ or in other words :

$$M'(t) = \underline{A}'(t)(M^t)^{-1}D/2.$$

Reducing δ if necessary, there exists a solution $M(t)$ of this differential equation on the interval $[-\delta, \delta]$, with an initial condition $M(0)$ satisfying $M(0)DM^t(0) = \underline{A}(0)$ (such an $M(0)$ exists provided the number of -1 on the diagonal on D is equal to the signature of $A(0)$). For such a solution, we see that the corresponding B is

$$DM^{-1}M' = DM^{-1}\underline{A}'(M^t)^{-1}D/2$$

and so it is symmetric. Then, the computation made earlier shows that $(MDM^t)' = \underline{A}'$. Since the equality $MDM^t = \underline{A}$ is satisfied at $t = 0$, it is thus satisfied for all $t \in [-\delta, \delta]$. \square

The second step consists of applying the vertical change of coordinates

$$\Theta : (q, p) \mapsto (q, p + du_q)$$

with $u(q) = \hat{q}^t B(q_1)\hat{q}/2$, so that $du_q = (\alpha(q), \hat{q}^t B(q_1))$ (where α is just some function of q that we do not need to compute). We obtain :

$$\begin{aligned} H \circ \Theta(q, p) &= H(q, 0) + v(q)(p_1 + \alpha(q)) - \hat{p}DB(q_1)\hat{q} + \frac{1}{2}a(q_1)(p_1 + \alpha(q))^2 \\ &\quad + \frac{1}{2}(\hat{p} + \hat{q}^t B(q_1))D(\hat{p}^t + B(q_1)\hat{q}) + O_3(\hat{q}, p) \\ &= H(q, 0) + v(q)\alpha(q) + \frac{1}{2}\hat{q}^t B(q_1)DB(q_1)\hat{q} + v(q)p_1 + a(q_1)p_1\alpha(q) + \frac{1}{2}a(q_1)p_1^2 \\ &\quad - \hat{p}DB(q_1)\hat{q} + \frac{1}{2}\hat{p}DB(q_1)\hat{q} + \frac{1}{2}\hat{q}^t B(q_1)D\hat{p}^t + \frac{1}{2}\hat{p}D\hat{p}^t + O_3(\hat{q}, p) \\ &= f(q) + w(q)p_1 + \frac{1}{2}a(q_1)p_1^2 + \frac{1}{2}\langle \hat{p}D, \hat{p} \rangle + O_3(\hat{q}, p), \end{aligned}$$

for some smooth functions f such that $f(q_1, 0) = H(0, 0)$, and w such that $w(q_1, 0) = 1$. Note in the above computation that $\hat{q}^t B(q_1)D\hat{p}^t = \hat{p}DB(q_1)\hat{q}$ because this is a 1×1 , hence obviously symmetric, matrix. \square

3. PERTURBING THE LINEARIZED MAPS.

We prove Theorem 1. We consider a periodic orbit θ of H , of minimal period T . We assume that there exists a neat time t_0 such that H is fiberwise isoenergetically non-degenerate at $\theta(t_0)$, and there is not loss of generality in assuming that $t_0 = 0$. We work locally near $x_0 = \theta(0)$, in the coordinates given by Theorem 5. In these coordinates, $x_0 = \theta(0) = (0, 0)$.

The Hamiltonian flow of H defines Poincaré transition maps along the orbit $(te_1, 0)$ between the sections $\{q_1 = 0\}$ and $\{q_1 = t\}$ for $|t| \leq \delta$. We shall be mostly interested in the restriction of this transition map to the energy level $\{H = 0\}$, called the restricted transition map. In the local coordinates, we have $dH(te_1, 0) = (0, e_1)$, hence the tangent space to the energy level along the orbit is $\{p_1 = 0\}$, and the tangent space of the restricted section $\{H = 0\} \cap \{q_1 = t\}$ is the space $\{q_1 = 0, p_1 = 0\}$, which we identify symplectically with \mathbb{R}^{2d} with coordinates $\hat{x} = (\hat{q}, \hat{p})$. The differential at 0 of the restricted transition map between the sections $\{q_1 = 0\}$ and $\{q_1 = t\}$ is then a symplectic linear map $L(t) \in Sp(2d)$.

Lemma 3. *Assume that $H : T^*\mathbb{R}^{d+1} \rightarrow \mathbb{R}$ satisfies the conclusions of theorem 5, then the restricted linearized transition maps $L(t)$ solve the differential equation*

$$\dot{L}(t) = Y(t)L(t),$$

where

$$Y(t) := \mathbb{J}\partial_{\hat{x}^2}^2 H(te_1, 0) = \begin{bmatrix} 0 & D \\ -K(t) & 0 \end{bmatrix},$$

$$K(t) := \partial_{\hat{q}^2}^2 H(te_1, 0), \quad D := \partial_{\hat{p}^2} H(te_1, 0).$$

Proof. Let X_H be the Hamiltonian vectorfield of H , and let $Z := X_H / \partial_{p_1} H$. In other words, Z is the reparametrisation of X_H which has its first coordinate equal to 1. The flow of Z defines the same local transition maps as X_H between the sections $\{q_1 = 0\}$ and $\{q_1 = t\}$, and this map is $(\hat{q}, \hat{p}) \mapsto \varphi_Z^t(0, \hat{q}, \hat{p})$ (the definition of Z ensures that this point does belong to the section $\{q_1 = t\}$).

The linearized system associated to X_H along the orbit $(te_1, 0)$ is given by the equations

$$q_1'(t) = (\partial^2 H / \partial^2 p_1)(te_1, 0)p_1 + (\partial^2 H / \partial q_1 \partial p_1)(te_1, 0)q_1 \quad , \quad p_1'(t) = 0,$$

$$\hat{x}'(t) = Y(t)\hat{x}(t) + \mathbb{J}(\partial^2 H / \partial q_1 \partial \hat{x})q_1 + \mathbb{J}(\partial^2 H / \partial p_1 \partial \hat{x})p_1.$$

Denoting $f = 1 / \partial_{p_1} H$, the expression $dZ(te_1, 0) = f dX(te_1, 0) + (e_1, 0)df(te_1, 0)$ shows that only the first equation in the linearized system of Z is different from the one of X_H . So the linearized system of Z along the orbit $(te_1, 0)$ is given by the equations

$$q_1'(t) = 0 \quad , \quad p_1'(t) = 0,$$

$$\hat{x}'(t) = Y(t)\hat{x}(t) + \mathbb{J}(\partial^2 H / \partial q_1 \partial \hat{x})q_1 + \mathbb{J}(\partial^2 H / \partial p_1 \partial \hat{x})p_1.$$

If $L(t)$ is the solution of the equation $L' = YL$ with initial condition $L(0) = Id$, then the differential of the flow φ_Z^t at 0 satisfies

$$(0, 0, \hat{x}) \mapsto (0, 0, L(t)\hat{x})$$

which implies that the differential at 0 of the transition map preserves the space $\{p_1 = 0\}$ and that its restriction to this subspace is given by $L(t)$. \square

We are now interested in the linearized transition map of the Hamiltonian $H + u$, where u is a smooth admissible potential, which here means that

$$u(te_1) = 0 \quad , \quad du(te_1) = 0 \quad \forall t \in [0, \delta].$$

If u is admissible and H satisfies the conclusions of theorem 5, then so does $H + u$, hence the linearized transition maps $L_u(t)$ associated to $H + u$ are described as follows:

Lemma 4. *If $H : T^*M \rightarrow \mathbb{R}$ satisfies the conclusions of theorem 5, then for an admissible potential u , the linearized transition maps $L_u(t)$ associated to $H + u$ satisfy*

$$(3.1) \quad \dot{L}_u(t) = Y_u(t)L_u(t).$$

$$\text{where } Y_u(t) = \mathbb{J}\partial_{\dot{x}^2}^2(H + u)(te_1, 0_n) = Y(t) + W_u(t), \quad W_u(t) = \begin{bmatrix} 0 & 0 \\ -\partial_{\dot{q}}^2 u(te_1) & 0 \end{bmatrix}.$$

In order to prove Theorem 1 we need to understand to what extent the linearized transition map $L_u(\delta)$ can be chosen by choosing the admissible potential u . Note that any compactly supported smooth curve $B(t) :]0, \delta[\rightarrow \mathcal{S}(d)$ is the curve $B_u(t)$ associated to some admissible potential u . So we are reduced to the following non-autonomous linear control problem:

$$(3.2) \quad \dot{L}_B(t) = Y(t)L_B(t) + \begin{bmatrix} 0 & 0 \\ B(t) & 0 \end{bmatrix} L_B(t),$$

where $Y(t)$ is a given curve of Hamiltonian matrices, and $B(t)$ is a control taking values in $\mathcal{S}(d)$. Recall that a matrix M is called Hamiltonian if $\mathbb{J}M$ is symmetric. We denote by $\mathfrak{sp}(2d)$ the set of Hamiltonian matrices of size $2d \times 2d$. This Lie algebra is the tangent space at the identity of the group $Sp(2d)$ of symplectic matrices. On this control problem, we can adapt [15] to our setting and get:

Proposition 6. *There exists a nowhere dense closed set $\mathcal{K}_D \subset \mathcal{S}(d)$, which depends on the diagonal matrix D , such that the differential at B_0 of the map*

$$C_c^\infty(]0, s[, \mathcal{S}(d)) \ni B \mapsto L_B(s) \in Sp(2d)$$

is onto for each $s \in]0, \delta[$ provided $K(0) + B_0(0) \notin \mathcal{K}_D$, where $C_c^\infty(]0, s[, \mathcal{S}(d))$ is the space of compactly supported smooth curves, and $L_B(t)$ is the solution starting with initial condition $L_B(0) = 0$ of the differential equation (3.2).

Assuming this Proposition, we finish the proof of the Theorem.

Proof of Theorem 1. Let U be an open set in the space C_θ^∞ of admissible potentials. There exists $u_0 \in U$ such that the corresponding $B_0(t) = \partial_{\dot{q}\dot{q}}^2 u_0(te_1, 0)$ satisfies $K(0) + B_0(0) \in \mathcal{K}_D$.

Proposition 6 then implies the existence of a finite dimensional linear subspace $F \subset C_c^\infty(]0, \delta[, \mathcal{S}(d))$ such that the restriction

$$F \ni B \mapsto L_{B_0+B}(\delta) \in Sp(2d)$$

is a C^1 submersion at $B = 0$. We denote $\mathcal{L}(B) := L_{B_0+B}(\delta)$ this map. There is a finite dimensional subspace E of smooth admissible potentials, the support of which do not intersect the remaining part $\pi \circ \theta(]0, \delta])$ of the projected orbit, and such that the map

$$E \ni u \mapsto B_u = \partial_{\dot{q}\dot{q}}^2 u \in F$$

is a linear isomorphism from E to F .

The restricted linearized Poincaré return map associated to the section $\{q_1 = 0\}$ for the Hamiltonian $H + u_0 + u$ is the product $OL(B_u)$, where O is the outer linearized transition map from the section $\{q_1 = \delta\}$ to the section $\{q_1 = 0\}$ associated to the Hamiltonian $H + u_0$ (or equivalently to $H + u_0 + u$). Since O does not depend on u , the map $u \mapsto OL(B_u)$ is a submersion in a neighborhood of 0 in E , which implies that the image of U contains an open set. \square

Proof of Proposition 6. Let us first recall some general theory, following [15]. We consider a smooth curve $Y(t)$ of Hamiltonian matrices, and the control problem

$$L'_B(t) = Y(t)L_B(t) + W(B(t))L_B(t),$$

where W is a linear map from a vector space E to $\mathfrak{sp}(2m)$ and $B(t)$ is a control taking values in E .

For each fixed $B \in E$, we define the following sequence of curves of matrices:

$$\begin{aligned} W_0(t, B) &\equiv W(B) \quad , \quad W_1(t, B) = [W_0(B), Y(t)], \\ W_{i+i}(t, B) &= \dot{W}_i(t, B) + [W_i(t, B), Y(t)]. \end{aligned}$$

Then, we consider the subspace

$$W_*E := Vect\{W_i(0, B), i \geq 0, B \in E\}.$$

The following general result is for example Proposition 2.1 in [15]:

Proposition 7. *If $W_*E = \mathfrak{sp}(2d)$, then for each $s > 0$, the differential at $B = 0$ of the map*

$$C_c^\infty([0, s[, E) \ni B \mapsto L_B(s) \in Sp(2d)$$

is onto. Here $L_B(s)$ is the solution at time s of the controlled differential equation for the given control $B(t)$, with the initial condition $L_B(0) = I$.

We now apply this general result to our situation of interest, where $E = \mathcal{S}(d)$ and $W(B) = \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}$, $Y(t) = \begin{bmatrix} 0 & D \\ -K(t) & 0 \end{bmatrix}$ with some fixed diagonal matrix D having diagonal elements equal to ± 1 . In order to deduce Proposition 6, it is sufficient to check that the assumption $W_*E = \mathfrak{sp}(2d)$ is satisfied under the assumption that $K(0)$ belongs to some open dense set \mathcal{K}_D . We compute:

$$\begin{aligned} W_0(t, B) &= W = \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}, \\ W_1(t, B) &= \begin{bmatrix} -DB & 0 \\ 0 & BD \end{bmatrix}, \\ W_2(t, B) &= \begin{bmatrix} 0 & -2DBD \\ -BDK(t) - K(t)DB & 0 \end{bmatrix}, \\ W_3(t, B) &= \begin{bmatrix} 3DBDK(t) + DK(t)DB & 0 \\ * & -BDK(t)D - 3KDBD \end{bmatrix}, \end{aligned}$$

where the $*$ bloc in W_3 is the derivative of the corresponding block in W_2 . We see that the matrices $W_0(0, B)$, $W_1(0, B)$ and $W_2(0, B)$ generate the space of matrices of the form $\begin{bmatrix} m_1 & m_2 \\ m_3 & -m_1^T \end{bmatrix}$, with Dm_1, m_2 and m_3 symmetric. In order that $W_*E = \mathfrak{sp}(2d)$, it is sufficient that the matrices $3DBDK(0) + K(0)DB$ generate a complement of $\mathcal{S}(d)$ in $\mathcal{M}(d)$, and this is equivalent to requiring that their antisymmetric parts $BDK(0) - K(0)DB$ generate the space $\mathcal{S}^-(d)$ of antisymmetric matrices. According

to the next Lemma, this holds provided $K(0)$ belongs to an appropriate open dense set \mathcal{K}_D , and this ends the proof of Proposition 6. \square

Lemma 5. *Let D be a fixed diagonal matrix with diagonal elements equal to ± 1 and let \mathcal{K}_D be the set of symmetric matrices $K \in \mathcal{S}(d)$ such that the map*

$$\mathcal{S}(d) \ni B \mapsto BDK - KDB \in \mathcal{S}^-(d)$$

is not onto. Then \mathcal{K}_D is a strict algebraic submanifold of $\mathcal{S}(d)$, its complement is thus a dense open set.

In the case where $D = I$ is the identity, \mathcal{K}_I is the space of symmetric matrices which have a multiple eigenvalue, see [7].

Proof. It is clear that $\mathcal{K}_D \subset \mathcal{S}(d)$ is an algebraic submanifold. It is thus enough to find one matrix K such that the map $B \mapsto BDK - KDB$ is onto. We claim that this conclusion holds if K is diagonal with distinct positive diagonal elements. In this case, $\Delta := DK = KD$ is also diagonal with distinct diagonal elements λ_i , and we compute $(BDK - KDB)_{ij} = (B\Delta - \Delta B)_{ij} = B_{ij}(\lambda_j - \lambda_i)$, which implies the claim. \square

4. PARAMETRIC TRANSVERSALITY

We deduce Theorem 2 from Theorem 1 using a variant of Abraham's parametric transversality principle, similarly to what is done in [4, 14]. Our implementation is slightly different and avoids the recurrence on the periods. We denote by $\varphi(t, x, u)$ the image of $x \in T^*M$ by the time t flow of $H + u$.

It is classical that 0 is a non-degenerate energy level for generic u . We recall a proof. Let $\mathcal{Z} \subset T^*M \times C^\infty(M)$ be the set of pairs (x, u) such that $d(H + u)(x) = 0$ and $(H + u)(x) = 0$. This is a closed set, and the projection of a closed set on the second factor is an F_σ (a countable union of closed sets). We use here that T^*M is a countable union of compact sets. So the set of potentials for which the 0 energy level is not regular is an F_σ . If u_0 is a potential, then Sard's theorem implies that there exist arbitrarily small constants a such that 0 is not a critical value of $H + u_0 + a$, which means that the set of potentials for which the 0 energy level is regular is dense. This ends the proof of this first step.

Next we introduce the subsets

$$\mathcal{P}^6 \subset \mathcal{P}^5 \subset \mathcal{P}^4 \subset \mathcal{P}^3 \subset \mathcal{P}^2 \subset \mathcal{P}^1 \subset]0, \infty[\times T^*M \times C^\infty(M)$$

defined as follows:

\mathcal{P}^1 is the set of triples (s, x, u) such that x is a s -periodic orbit of $H + u$ and $(H + u)(x) = 0$, it includes $]0, \infty[\times \mathcal{Z}$. The set \mathcal{P}^1 is closed.

\mathcal{P}^2 is the subset of triples such that s is the minimal period of x (hence \mathcal{P}^2 is disjoint from $]0, \infty[\times \mathcal{Z}$).

\mathcal{P}^3 is the subset of triples such that, in addition, 0 is a neat point of the orbit of x for $H + u$.

\mathcal{P}^4 is the subset of triples such that, in addition, H is fiberwise isoenergetically non-degenerate at x .

\mathcal{P}^5 is the subset of triples (s, x, u) such that, in addition, the $(H + u)$ -orbit of x is non-degenerate as an s -periodic orbit (but not necessarily as an ns -periodic orbit, $n \geq 2$).

\mathcal{P}^6 is the subset of triples such that, in addition, the restricted linearized return map of the $(H + u)$ -orbit of x does not belong to Υ .

Lemma 6. *The inclusions $\mathcal{P}^5 \subset \mathcal{P}^4 \subset \mathcal{P}^3 \subset \mathcal{P}^2 \subset \mathcal{P}^1$ are open. As a consequence the differences $\mathcal{P}^i - \mathcal{P}^{i+1}$, $1 \leq i \leq 4$ are locally closed, hence F_σ .*

Recall that a set is called locally closed if it is the intersection of a closed and of an open set.

Proof. The topology on $C^\infty(M)$ is metrizable and separable. The metrizability implies that all locally closed sets are F_σ .

To prove that $\mathcal{P}^1 - \mathcal{P}^2$ is closed we consider a sequence $(s_k, x_k, u_k) \rightarrow (s, x, u)$ of iterated orbits converging to $(s, x, u) \in \mathcal{P}^1$. We denote by S_k the minimal period of x_k , so that $s_k = i_k S_k$ for some integers $i_k \geq 2$. By taking a subsequence, we can assume that i_k either is constant or converges to $+\infty$. In the first case, denoting by $i \geq 2$ the constant value of the sequence i_k , we obtain that $S_k \rightarrow s/i$, and that x is s/i -periodic, so that s is not the minimal period of x . In the second case, we obtain that $x \in \mathcal{Z}$. In both cases, (s, x, u) does not belong to \mathcal{P}^2 .

To prove that $\mathcal{P}^2 - \mathcal{P}^3$ is closed in \mathcal{P}^2 , we consider a sequence (s_k, x_k, u_k) in $\mathcal{P}^2 - \mathcal{P}^3$ converging to a limit (s, x, u) in \mathcal{P}^2 . We denote by $Q_k(t)$ the projected $(H+u_k)$ -orbit of x_k . Since 0 is not a neat time we can assume by taking a subsequence that either $\dot{Q}_k(0) = 0$ for each k or there exists times $t_k \in]-s_k/2, 0[\cup]0, s_k/2]$ such that $Q_k(0) = Q_k(t_k)$. The first case immediately implies that $\dot{Q}(0) = 0$, where $Q(t)$ is the limit projected orbit. In the second situation, if the sequence t_k has an accumulation point $t \neq 0$ then $Q(t) = Q(0)$, and since s is the minimal period of x this implies that 0 is not a neat time at the limit. Otherwise $t_k \rightarrow 0$. Then the equation $Q_k(t_k) = Q_k(0)$ imply at the limit that $\dot{Q}(0) = 0$, hence once again 0 is not a neat time at the limit.

The other claims are clear. \square

Let us denote by Π the projection $]0, \infty[\times T^*M \times C^\infty(M) \rightarrow C^\infty(M)$. We are interested in proving that $\Pi(\mathcal{P}^4 - \mathcal{P}^6)$ is a nowhere dense F_σ . This follows from Proposition 8 and Proposition 9 below. Since the fiber $]0, \infty[\times T^*M$ is a countable union of compact sets, the Π -image of an F_σ is an F_σ , this can be applied to $\mathcal{P}^4 - \mathcal{P}^5$ and to $\mathcal{P}^5 - \mathcal{P}^6$.

Proposition 8. *The set $\Pi(\mathcal{P}^5 - \mathcal{P}^6) \subset C^\infty(M)$ is a nowhere dense F_σ .*

Proof. The continuous dependence of non-degenerate periodic orbits imply that the restriction of Π to \mathcal{P}^5 is a local homeomorphism onto an open subset of $C^\infty(M)$: Each point (s, x, u) has a neighborhood \mathcal{P}_{loc}^5 in \mathcal{P}^5 in restriction to which the map Π is an homeomorphism onto an open subset of $C^\infty(M)$. We can take a smooth section at x of the vectorfield, with symplectic coordinates. By reducing \mathcal{P}_{loc}^5 if necessary, we can assume that the first return map is well defined inside the section for each point of the neighborhood, its differential $L(s, x, u)$ is a continuous map with values in $Sp(2d)$. Theorem 1 implies that this map is weakly open (the image of a non-empty open set contains a non-empty open set). As a consequence, $\mathcal{P}_{loc}^5 - \mathcal{P}^6 = \mathcal{P}_{loc}^5 \cap L^{-1}(\Upsilon)$ is a nowhere dense F_σ . Its image under the homeomorphism $\Pi|_{\mathcal{P}_{loc}^5}$ is then a nowhere dense F_σ in $C^\infty(M)$. Finally, since \mathcal{P}^5 is a separable metric space, we can cover it with **countably** many open sets \mathcal{P}_{loc}^5 satisfying this conclusion. We deduce that $\Pi(\mathcal{P}^5 - \mathcal{P}^6)$ is a countable union of nowhere dense F_σ , hence it is a nowhere dense F_σ . \square

Proposition 9. *The set $\Pi(\mathcal{P}^4 - \mathcal{P}^5) \subset C^\infty(M)$ is a nowhere dense F_σ .*

Proof. Since $]0, \infty[\times T^*M \times C^\infty(M)$ is a separable metric space, it is enough to prove the property locally, more precisely it is enough to prove that each point $(s_0, x_0, u_0) \in \mathcal{P}^4$ has an open neighborhood \mathcal{P}_{loc}^4 such that $\Pi(\mathcal{P}_{loc}^4 - \mathcal{P}^5)$ is nowhere dense. Note that $\mathcal{P}_{loc}^4 - \mathcal{P}^5$ is locally closed, hence its projection is an F_σ .

We can assume without loss of generality that $u_0 = 0$, and work in the local coordinates near x_0 given by Theorem 5. In these coordinates, $x_0 = 0$. We consider the transverse section $\{q_1 = 0\}$, and its intersection with the energy surface $\{H = 0\}$. The tangent space at $x = 0$ to this energy surface is $\{p_1 = 0\}$, and this implies that $\hat{x} = (\hat{q}, \hat{p})$ are symplectic local coordinates of the restricted section

$$\Lambda(u) := \{(q, p) \in \mathbb{R}^{2(d+1)} : q_1 = 0, (H + u)(q, p) = 0\}$$

provided u belongs to a sufficiently small neighborhood $C_{loc}^\infty(M)$ of 0. We denote by $x(\hat{x}, u)$ the point of $\Lambda(u)$ which has coordinate \hat{x} , by

$$\tau(\hat{x}, u) : \mathbb{R}_{loc}^{2d} \times C_{loc}^\infty(M) \longrightarrow \mathbb{R}$$

the first return time of the point $x(\hat{x}, u)$, and by

$$\psi(\hat{x}, u) : \mathbb{R}_{loc}^{2d} \times C_{loc}^\infty(M) \longrightarrow \mathbb{R}^{2d}$$

the return map expressed in coordinates. So $\psi(\hat{x}, u)$ is the (\hat{q}, \hat{p}) coordinates of the point $\varphi(\tau(\hat{x}, u), x(\hat{x}, u), u)$. Let

$$\mathcal{Y} \subset \mathbb{R}_{loc}^{2d} \times C_{loc}^\infty(M)$$

be the set of solutions of the equation $\psi(\hat{x}, u) = \hat{x}$, and let $\mathcal{Y}^5 \subset \mathcal{Y}$ be the set of non-degenerate solutions of this equation, meaning those such that $\partial_{\hat{x}}\psi(\hat{x}, u)$ does not have the eigenvalue 1. Given $(\hat{x}, u) \in \mathbb{R}_{loc}^{2d} \times C_{loc}^\infty(M)$, we have $(\hat{x}, u) \in \mathcal{Y}$ if and only if $(\tau(\hat{x}, u), x(\hat{x}, u), u) \in \mathcal{P}^1$, and this is also equivalent to $(\tau(\hat{x}, u), x(\hat{x}, u), u) \in \mathcal{P}^4$ (because \mathcal{P}^4 is open in \mathcal{P}^1). We have $(\hat{x}, u) \in \mathcal{Y}^5$ if and only if $(\tau(\hat{x}, u), x(\hat{x}, u), u) \in \mathcal{P}^5$. The set

$$\mathcal{P}_{loc}^4 := \mathcal{P}^4 \cap \{(\tau(\hat{x}, u), \varphi(t, x(\hat{x}, u), u), u) : t \in \mathbb{R}, (\hat{x}, u) \in \mathcal{Y}\}$$

is an open neighborhood of $(s_0, 0, 0)$ in \mathcal{P}^4 , which has the property that $\Pi(\mathcal{P}_{loc}^4) = \Pi(\mathcal{Y})$ and

$$\Pi(\mathcal{P}_{loc}^4 - \mathcal{P}^5) = \Pi(\mathcal{Y} - \mathcal{Y}^5)$$

where we still use Π for the projection on the second factor in the product $\mathbb{R}_{loc}^{2d} \times C_{loc}^\infty(M)$.

So we are reduced to proving that $\Pi(\mathcal{Y} - \mathcal{Y}^5)$ is nowhere dense. The general idea is that \mathcal{Y} is a submanifold in $\mathbb{R}_{loc}^{2d} \times C_{loc}^\infty(M)$ and that \mathcal{Y}^5 is the set of regular points of Π on this manifold, which allows to conclude using Sard's theorem. In order to avoid manipulating differential calculus on the Fréchet space $C^\infty(M)$, we restrict ourselves to a finite dimensional subspace.

Lemma 7. *For any neighborhood U of q_0 in M , there exists a finite dimensional subspace $E \subset C^\infty(M)$ formed by potentials supported inside U and null on the orbit, such that $\partial_u\psi(0, 0)$ sends E onto \mathbb{R}^{2d} .*

We finish the proof of Proposition 9 assuming the Lemma. By continuity, the equality $\partial_u\psi(\hat{x}, u) \cdot E = \mathbb{R}^{2d}$ holds for all $(\hat{x}, u) \in \mathbb{R}_{loc}^{2d} \times C_{loc}^\infty(M)$ (we decrease the neighborhoods if necessary). Then for each $v \in C_{loc}^\infty(M)$, the map

$$\Psi : \mathbb{R}_{loc}^{2d} \times E_{loc} \ni (\hat{x}, u) \mapsto \Psi(\hat{x}, u) := \psi(\hat{x}, v + u) - \hat{x} \in \mathbb{R}^{2d}$$

is a submersion, provided E_{loc} is a sufficiently small neighborhood of 0 in E . As a consequence, the set $N := \Psi^{-1}(0)$ is a submanifold, and the points $(\hat{x}, u) \in N$ such that $\partial_{\hat{x}}\Psi(\hat{x}, u)$ is not invertible are the singular points of the projection $\Pi|_N$ (this follows from elementary considerations in finite dimensional linear algebra). If $v + u$ belongs to $\Pi(\mathcal{Y} - \mathcal{Y}^5)$, then there exists a point $\hat{x} \in \mathbb{R}_{loc}^{2d}$ such that $(\hat{x}, v + u) \in \mathcal{Y} - \mathcal{Y}^5$, which implies that $(\hat{x}, u) \in N$ is a critical point of $\Pi|_N$. So u has to be a critical value of $\Pi|_N$. By Sard's Theorem, there exist regular values of $\Pi|_N$ arbitrarily close to 0, and this implies that v does not belong to the interior of $\Pi(\mathcal{Y} - \mathcal{Y}^5)$. Since this holds for all $v \in C_{loc}^\infty(M)$, we conclude that $\Pi(\mathcal{Y} - \mathcal{Y}^5)$ is nowhere dense. This ends the proof of Proposition 9. \square

Proof of Lemma 7. As for the proof of Lemma 2 in [4], the key element is that the matrix $D = \partial_{\hat{p}\hat{p}}^2 H(0)$ is invertible, which is precisely the expression in coordinates of fiberwise isoenergetic non-degeneracy. We work in the local coordinates near $x = 0$ given by Theorem 5. As in the proof of Lemma 3, we consider the vectorfield $Z(x, u) := \mathbb{J}\partial_x(H + u)(x)/\partial_{p_1}H(x)$ and denote by $\varphi_Z(t, x, u)$ its flow. The curve $\omega(t) := \partial_u\varphi_Z(te_1, 0, 0) \cdot u$ solves the differential equation

$$\omega'(t) = \partial_x Z(te_1, 0, 0) \cdot \omega(t) + \partial_u Z(te_1, 0, 0) \cdot u.$$

Note that $\omega(t)$ depends (linearly) on u but we do not mention this explicitly to simplify notations. Using that $\partial_{p_1}H(te_1, 0, 0) = 1$, we obtain

$$\partial_x Z(te_1, 0, 0) = \begin{bmatrix} 0 & & 0 \\ & 0 & D \\ 0 & [-K(t) & 0] \end{bmatrix}, \quad \partial_u Z(te_1, 0, 0) \cdot u = \begin{bmatrix} 0 \\ -\partial_{q_1}u(te_1) \\ 0 \\ -\partial_{\hat{q}}u(te_1) \end{bmatrix},$$

where we have used bloc decomposition associated to the coordinates $x = (x_1, \hat{x}) = (x_1, (\hat{q}, \hat{p})) = (q_1, p_1, \hat{q}, \hat{p})$. The equations for the x_1 and \hat{x} coordinates of ω are thus uncoupled, and, denoting by $\hat{\omega}$ the latter, we get

$$(4.1) \quad \hat{\omega}'(t) = Y(t)\hat{\omega}(t) + \hat{w}_u(t),$$

where $Y(t) = \begin{bmatrix} 0 & D \\ -K(t) & 0 \end{bmatrix}$ and $\hat{w}_u(t) = \begin{bmatrix} 0 \\ -\partial_{\hat{q}}u(te_1) \end{bmatrix}$. We denote by $\psi(t, \hat{x}, u)$ the restricted transition map between the sections $\{q_1 = 0\}$ and $\{q_1 = t\}$, so that $\psi(t, \hat{x}, u)$ is the \hat{x} -coordinate of $\varphi_Z(t, x(\hat{x}, u), u)$ where $x(\hat{x}, u)$ is the point of $\Lambda(u)$ with coordinate \hat{x} . As a consequence,

$$\partial_u\psi(t, 0, 0) \cdot u = \hat{\omega}(t)$$

is determined by the differential equation (4.1), and by the initial condition $\hat{\omega}(0) = 0$.

We are thus once more reduced to a linear controlled differential equation :

$$\hat{\omega}'(t) = Y(t)\hat{\omega}(t) + b(t)$$

where $b(t)$ is a control with values in $\{0\} \times \mathbb{R}^d$. Denoting by $M(t)$ the curve of matrices such that $M(0) = I$ and $M'(t) = Y(t)M(t)$ and by $y(t) = M^{-1}(t)\hat{\omega}(t)$, this controlled equation is reduced to :

$$y'(t) = M^{-1}(t)b(t).$$

We fix $\sigma > 0$ and consider a control of the form $b(t) = \delta(t)\alpha$, with α fixed in $\{0\} \times \mathbb{R}^d$ and where $\delta(t)$ is a smooth approximation of the dirac at 0 supported in $]0, \sigma[$.

We see that the corresponding $y(s)$ is an approximation of α . Now for a control $b(t) = \delta'(t)\beta$, we have

$$y'(t) = (M^{-1}\delta)' \beta - (M^{-1})' \delta \beta = (M^{-1}\delta)' \beta + M^{-1}Y\delta\beta$$

hence $y(\sigma) \approx Y(0)\beta$. Since D is invertible, the vectors $e_i + Y(0)e_j$, $1 \leq i, j \leq d$ span \mathbb{R}^{2d} (where e_i is the standard base of \mathbb{R}^d). So if δ is a sufficiently good approximation of the Dirac at 0, then the values $y(\sigma)$ of the solutions of the controlled differential equation corresponding to the controls $\delta(t)e_i + \delta'(t)e_j$, $1 \leq i, j \leq d$, span \mathbb{R}^{2d} . Now for each i, j , we consider a potential u_{ij} supported in U , null on the orbit, and such that $\hat{w}_{u_{ij}} = \delta(t)e_i + \delta'(t)e_j$, $1 \leq i, j \leq d$ and define E as the space generated by these potentials. We can moreover assume that the supports are disjoint from the remaining part $Q([\sigma, s_0])$ of the projected orbit. The linear map $\partial_u \psi(\sigma, 0, 0) : u \mapsto \hat{w}(\sigma)$ is then onto.

To end the proof, we denote by $G(\sigma, \hat{x})$ the restricted section map from $\{q_1 = \sigma\}$ to $\{q_1 = 0\}$ along the periodic orbit. This map does not depend on $u \in E$ since the supports have been taken disjoint from $Q([\sigma, s_0])$. The restricted Poincaré return map satisfies

$$\psi(\hat{x}, u) = G(\sigma, \psi(\sigma, \hat{x}, u))$$

and $\partial_u \psi(0, 0) = \partial_{\hat{x}} G(\sigma, 0) \circ \partial_u \psi(\sigma, 0, 0)$ is onto. \square

5. ORBITS INSIDE A SUBMANIFOLD.

We prove Theorem 3. It is enough to prove the statement under the assumption that Σ is a submanifold of positive codimension transverse to the vertical. Let $\mathcal{R}(\epsilon)$ be the set of couples $(x, u) \in T^*M \times C^\infty(M)$ such that $\partial_p H(x) \neq 0$ and such that $\varphi([- \epsilon, \epsilon] \times \{x\} \times \{u\}) \subset \Sigma$. We want to prove that $\Pi(\cup_{\epsilon > 0} \mathcal{R}(\epsilon)) = \cup_{\epsilon > 0} \Pi(\mathcal{R}(\epsilon))$ is a nowhere dense F_σ . Note that $\mathcal{R}(\epsilon)$ is locally closed, hence each $\Pi(\mathcal{R}(\epsilon))$, $\epsilon > 0$ is an F_σ , hence so is the union.

It is enough to prove that each of the sets $\Pi(\mathcal{R}(\epsilon))$ is nowhere dense, and as above, it is even enough to prove this locally near a point $(x_0, u_0) \in \mathcal{R}(\epsilon)$. More precisely, given $(x_0, u_0) \in \mathcal{R}(\epsilon)$ it is enough to prove that there exists a neighborhood $(T^*M)_{loc}$ of x_0 in T^*M such that $\Pi(((T^*M)_{loc} \times C^\infty(M)) \cap \mathcal{R}(\epsilon))$ is nowhere dense.

We consider small times $0 < \sigma_0 < \sigma_1 < \sigma_2 < \dots < \sigma_k < \sigma_{k+1} < \epsilon$, with $k > 2d + 2$ and define the map

$$\Phi : T^*M \times C^\infty(M) \ni (x, u) \mapsto (\varphi(\sigma_1, x, u), \dots, \varphi(\sigma_k, x, u)) \in (T^*M)^k,$$

so that $\mathcal{R}(\epsilon) \subset \Phi^{-1}(\Sigma^k)$.

Lemma 8. *For each (x_0, u_0) such that $\partial_p H(x_0) \neq 0$, the times σ_i can be chosen such that there exists a finite dimensional subspace $E \subset C^\infty(M)$ for which the map*

$$T^*M \times E \ni (x, u) \mapsto \Phi(x, u_0 + u)$$

is transverse to Σ^k at the point $(x_0, 0)$.

Assuming the Lemma, there exist neighborhoods $(T^*M)_{loc}$ and $C_{loc}^\infty(M)$ of x_0 and u_0 such that the map

$$T^*M \times E \ni (x, u) \mapsto \Phi(x, u_1 + u)$$

is transverse to Σ^k at $(x_1, 0)$ provided $(x_1, u_1) \in (T^*M)_{loc} \times C_{loc}^\infty(M)$. This implies that $\Phi^{-1}(\Sigma^k)$ is a submanifold in $(T^*M)_{loc} \times E_{loc}$, of codimension equal to the codimension of Σ^k in $(T^*M)^k$, and this codimension is at least k . Since $k > 2d + 2 =$

$\dim(T^*M)$, the projection of $\Phi^{-1}(\Sigma^k)$ on E is nowhere dense. As a consequence, there exists arbitrarily small $u \in E$ such that u does not belong to $\Pi(\Phi^{-1}(\Sigma^k))$. For such u , there is no $x \in (T^*M)_{loc}$ such that $\Phi(x, u_1 + u) \in \Sigma^k$, which implies that there is no $x \in (T^*M)_{loc}$ such that $(x, u_1 + u) \in \mathcal{R}(\epsilon)$. We have proved that $\Pi(((T^*M)_{loc} \times C^\infty(M)) \cap \mathcal{R}(\epsilon))$ is nowhere dense, and this implies Theorem 3. \square

Proof of Lemma 8. There is no loss of generality to assume that $u_0 = 0$. Since H is not assumed fiberwise isoenergetically non-degenerate at x_0 , Theorem 5 can't be applied at x_0 . However there exist local coordinates such that $\varphi(t, x_0, 0) = (te_1, 0)$ for small $|t| < \delta$ for some $\delta > 0$. A look at the proof of Theorem 5 shows that such coordinates exist under the only assumption that $\partial_p H(x_0) \neq 0$. We assume that $\sigma_i \in]0, \delta[$, in particular the projected orbit is one to one on $[0, \sigma_{k+1}]$.

Recall that, for a given potential u , the curve $t \mapsto d(t) := \partial_u \varphi(t, 0, 0) \cdot u$ is determined by the non-homogeneous linear equation

$$d'(t) = \xi(t)d(t) + (0, -du(te_1))$$

and by the initial condition $d(0) = 0$, where $\xi(t) = \mathbb{J}\partial_{xx}^2 H(te_1, 0)$ is the linearized equation. Contrary to the earlier sections of the paper, we have no information on $\xi(t)$. We denote by Ξ_s^t the family of solutions of the equation

$$\partial_t \Xi_s^t = \xi(t)\Xi_s^t$$

with $\Xi_s^s = I$. Observe that $\Xi_s^t = \partial_x \varphi(t - s, se_1, 0)$.

Let $\delta_i(t)$ be smooth approximations of the Dirac function at time σ_i , supported in $]\sigma_{i-1}, \sigma_i[$. For each i and j , there exists a smooth potential $u_{i,j}$ such that

$$du_{i,j}(te_1) = e_j \delta_i(t),$$

where (e_j) is the standard base of \mathbb{R}^{d+1} . The differential $\partial_u \varphi(\sigma_i, 0, 0) \cdot u_{i,j}$ is approximately the vertical vector $l_j := (0, e_j) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$. Then for $i' \geq i$,

$$\partial_u \varphi(\sigma_{i'}, 0, 0) \cdot u_{i,j} \approx \Xi_{\sigma_i}^{\sigma_{i'}} l_j.$$

As a consequence, we have

$$\partial_u \Phi(0, 0) \cdot (u_{i,j}) \approx \eta_{i,j} := (0, \dots, 0, l_j, \Xi_{\sigma_i}^{\sigma_{i+1}} l_j, \dots, \Xi_{\sigma_i}^{\sigma_k} l_j) \in (\mathbb{R}^{2d+2})^k.$$

The vectors $\eta_{i,j}$ span the product of verticals $(\{0\} \times \mathbb{R}^{d+1})^k$ which is a subspace transverse to Σ^k . By taking for δ_i sufficiently good approximations of Dirac functions, we can make the vectors $\partial_u \Phi(0, 0) \cdot u_{i,j}$ as close as we want to $\eta_{i,j}$, and then they also span a vector subspace transverse to Σ^k . As a consequence the $k(d+1)$ -dimensional vector space E generated by the potentials $u_{i,j}$ has the property that $\partial_u \Phi(0, 0)$ sends E to a subspace transverse to Σ^k . \square

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PSL RESEARCH UNIVERSITY, ÉCOLE NORMALE SUPÉRIEURE, DMA , RUE D'ULM 75230 PARIS CEDEX 05, FRANCE

Email address: shahriar.aslani@ens.fr

PSL RESEARCH UNIVERSITY, UNIVERSITÉ PARIS DAUPHINE, CEREMADE, PLACE DU MARÉCHAL DE LATTRE DE TASSIGNY, 75775 PARIS CEDEX 16, FRANCE

Email address: pbernard@ceremade.dauphine.fr