PhD Thesis Defense: Bumpy metric theorem in the sense of Mañé for non-convex Hamiltonian vector fields

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Overview

We study Hamiltonians that are defined on cotangent bundle of a smooth manifold.

One of the main themes in our research is to perturb a Hamiltonian $H : T^{*}M \to \mathbb{R}$ via adding a smooth function $u : M \to \mathbb{R}$.

Such a function that depends only on the base manifold is called a potential.

Overview

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Such a function that depends only on the base manifold is called a potential.

This notion of perturbing Hamiltonians is suggested by Mañé [Mn96].

Definition

A property (p) is called Mañé-generic if, given a smooth Hamiltonian $H : T^*M \to \mathbb{R}$; there exists a G_δ dense subset $G \subset C^\infty(M)$ such that for each $u \in G$; $H + u$ admits the property (p).

Objective

Objective: The main goal of the thesis is to know that, given a smooth non-convex Hamiltonian H and $k \in \mathbb{R}$, to what extent all closed orbits in $(H+u)^{-1}(k)$ are non-degenerate where u is a generic smooth potential.

- A periodic orbit is non-degenerate if its associated linearized Poincaré map does not take roots of unity as an eigenvalue.
- A Hamiltonian $H(q,p) : T^*M \to \mathbb{R}$ is convex whenever $\partial^2_{p^2} H(q,p)$ is positive-definite for all $(q,p) \in T^*M$.

Bumpy metric theorem

A Riemannian metric that all its closed geodesics are non-degenerate is called bumpy.

Theorem (Abraham [Abr70], Klingenberg and Takens [KT72], and Anosov [Ano83])

Let $R^r(M)$; r 2; be the set of all C^r Riemannian metrics on a smooth manifold M: Define

$$B^r(M) := \{g \; 2 \; R^r(M) \; j \text{ all closed geodesics of } g \text{ are non-degenerate}\}:$$

as the set of C^r bumpy Riemannian metrics on M: Then, $B^r(M)$ is a G dense subset of $R^r(M)$:

Mañé perturbation and conformal perturbation of metrics

A smooth conformal perturbation of a Riemannian metric $g\ 2\ R^r(M)$ is a perturbation of the form $e^{b(q)}g$ where $b(q)\ 2\ C^{1}(M)$.

A C^{1}-small conformal perturbation of a Riemannian metric is equivalent to perturbing its associated Hamiltonian via adding a C^{1}-small potential.

Bumpy metric theorem a la Mañé for convex Hamiltonians

Theorem (Bumpy metric theorem a la Mañé for convex Hamiltonians)

Let M be a $(d+1)$-dimensional smooth manifold where $d \geq 1$. Assume that a smooth convex Hamiltonian $H : T^*M \to \mathbb{R}$ is given. For a given $k \in \mathbb{R}$, there exists a G_δ dense subset $\mathcal{G} \subset C^\infty(M)$ such that for each $u \in \mathcal{G}$, $(H+u)^{-1}(k)$ is a regular energy level and all periodic orbits in $(H+u)^{-1}(k)$ are non-degenerate.

- Oliveira [Oli08] studied the theorem for $d = 1$. A machinery to study linearized perturbed Poincaré maps was missing to extend Oliveira's result to higher dimensions.
- Using geometric control methods, Rifford and Ruggiero [RR12] achieves the missing block of the proof.

Perturbation theorem

Theorem (Rifford and Ruggiero, 2012)

Let M be a $(d+1)$-dimensional smooth manifold where $d \geq 1$. Suppose that $\gamma(t) \in H^{-1}(k)$ is a non-trivial periodic orbit of a smooth convex Hamiltonian $H : T^*M \to \mathbb{R}$. Consider

$$P_u(\Sigma, \gamma) : \Sigma \cap (H+u)^{-1}(k) \to \Sigma \cap (H+u)^{-1}(k), \quad u \in C^\infty_\gamma(M),$$

as the Poincaré map with respect to $\gamma(t)$ and Hamiltonian vector field of $H+u$, where Σ is a transverse section to $\gamma(t)$. The map $F(\Sigma, \gamma; H+u)$ defined as

$$C^\infty_\gamma(M) \ni u \mapsto dP_u \in Sp(2d)$$

is weakly open.

$C^\infty_\gamma(M)$ which is defined as follows, is the set of admissible potentials with respect to an orbit $\gamma(t)$

$$C^\infty_\gamma(M) := \{u \in C^\infty(M) \mid u(\gamma(t)) = 0, \ du(\gamma(t)) = 0, \ \text{for all } t \in \mathbb{R}\}.$$

Perturbation theorem

Theorem (Rifford and Ruggiero, 2012)

Let M be a $(d+1)$-dimensional smooth manifold where d 1. Suppose that (t) 2 H $^{1}(k)$ is a non-trivial periodic orbit of a smooth convex Hamiltonian $H : T$ M ! $\mathbb{R}$: Consider

$$P_u(\;;\;) : \quad \backslash\ (H+u)\ ^{1}(k)\ !\quad \backslash\ (H+u)\ ^{1}(k); \quad u\ 2\ C^{1}(M);$$

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is weakly open.

A mapping is called weakly open if the image of a non-empty open set has a non-empty interior.

Perturbation theorem

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$$P_u(\; ; \quad) : \quad \cap (H+u)^{\;1}(k) \to \quad \cap (H+u)^{\;1}(k); \quad u \in C^{1}(M);$$

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It has been proven by Lazrag, Rifford, and Ruggiero [LRR16] that $F(\; ; H+u)$ is open.

Structure of the presentation

We first introduce a more general setting in which we can study similar problems for non-convex Hamiltonians. This setting is consistent with previous results for the convex case, so it allows us to explain the contribution of the thesis to convex version of the bumpy metric theorem as well.

Fiberwise iso-energetically non-degeneracy at a point

Definition

A smooth Hamiltonian $H : T^*M \to \mathbb{R}$ is *fiberwised iso-energetically non-degenerate* at a point $(q, p) \in T^*M$ if

$$\det \begin{bmatrix} \partial^2_{p^2} H(q;p) & \partial_p H(q;p) \\ [\partial_p H(q;p)]^T & 0 \end{bmatrix} \neq 0.$$

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For a given smooth Hamiltonian $H : T^*M \to \mathbb{R}$, we define

$${}_H := \left\{ (q,p) \in T^*M \mid \det \begin{bmatrix} \partial^2_{p^2} H(q,p) & \partial_p H(q,p) \\ [\partial_p H(q,p)]^T & 0 \end{bmatrix} = 0 \right\}.$$

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A smooth Hamiltonian $H : T \ M \to \mathbb{R}$ is *fiberwised iso-energetically non-degenerate* at a point $(q, p) \in T \ M$ if

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Note that whenever H is convex, $(q, p) \in \ _H$ if and only if $\partial_p H(q, p) = 0.$ For a convex Hamiltonian $H,$ exactly one point per fiber belongs to $\ _H.$

Bumpy metric theorem in the sense of Mañé for non-convex Hamiltonians

Hypothesis 1

The subset $\Sigma_H \subset T^*M$ is contained in a countable union of manifolds of positive codimension which are transversal to the vertical fibrations.

Theorem

Assume that $H : T^*M \to \mathbb{R}$ is a smooth Hamiltonian such that Σ_H satisfies Hypothesis 1. For a given $k \in \mathbb{R}$, there exists a G_δ dense subset $\mathcal{G} \subset C^\infty(M)$ such that for each $u \in \mathcal{G}$, $(H+u)^{-1}(k)$ is a regular energy level and all periodic orbits in $(H+u)^{-1}(k)$ that are admitting a neat time are non-degenerate.

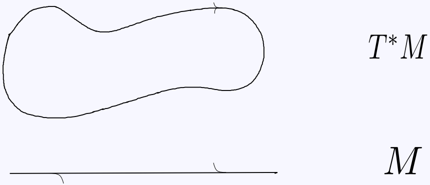
Symmetric orbits

Definition

Suppose $(t) = \big(Q(t); P(t)\big)$ is a periodic orbit with minimum period of a Hamiltonian vector field. We say time t_0 is a *neat time* for (t) if $Q(t_0) \neq 0;$ and there is not exists a time $s \neq t_0$ such that $Q(t_0) = Q(s)$ modulo $:$
If t_0 is a neat time for $(t);$ then we call (t_0) a neat point.

Definition

An orbit of a Hamiltonian vector field is called *symmetric* if it does not admit any neat time.



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Definition

An orbit of a Hamiltonian vector field is called *symmetric* if it does not admit any neat time.

Kozlov [Koz76] shows that, under some conditions, reversible Hamiltonians of the form $H(q; p) = g_q(p) + u(q)$ —where g is a Riemannian metric and u is a potential— are admitting a symmetric orbit.

Methods of proof

Now we discuss about some of the methods that we used in the proof of the bumpy metric theorem in the sense of Mañé for non-convex Hamiltonians.

Normal form

Theorem

Suppose $\underline{H}(q,p) : T \; \mathbb{R}^{d+1} \to \mathbb{R}$ *is a smooth Hamiltonian. Consider* $\dot{}(t) = \big(\underline{Q}(t), \underline{P}(t)\big) \in \underline{H}^{\;-1}(0)$ *as an orbit of Hamiltonian vector field of* $\underline{H}$*: Assume that* $\dot{}(0) \notin \;\; {}_{\underline{H}}$ *. There exist* $\; > 0$*, a smooth* *fibered symplectomorphism* $\; : T \; \mathbb{R}^{d+1} \to T \; \mathbb{R}^{d+1}$*, and a smooth function* $z(q) : \mathbb{R}^{d+1} \to \mathbb{R}$ *such that* $\big(Q(t), P(t)\big) := \;^{-1}(\dot{})$ *is an orbit of Hamiltonian vector field of* $H := z(q)(\underline{H} \quad)$*, and for all* $t \in [\;\; , \;\;]$ *we have*

(1) $Q(t) = te_1, \quad e_1 = (1, 0_d)$

(2) $P(t) = 0$

$$\left.\begin{array}{ll}(3) & \partial^2_{p_1 p} H(te_1, 0) = 0, \\ (4) & \partial^2_{qp} H(te_1, 0) = 0, \\ (5) & \partial^2_{p^2} H(te_1, 0) = D,\end{array}\right\} \Rightarrow \partial^2_{x^2} H(te_1, 0) = \begin{bmatrix} 0 & 0 & & 0 \\ 0 & K(t) & & \\ & & & 0 \\ & 0 & 0 & D \end{bmatrix}$$

where D *is a constant diagonal matrix with only 1 and -1 entries.*

Remarks about the normal form

A symplectomorphism $: T\mathbb{R}^{d+1} \to T\mathbb{R}^{d+1}$ is fibered if it preserves the vertical fibrations.

A fibered symplectomorphism $(q;p) : T\mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is homogeneous if it preserves the zero section. In other words, is homogeneous if and only if

$$(q;p) = \big(\varphi(q); [d\varphi^{\;1}(q)]^T p\big);$$

where $\varphi : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is a diffeomorphism.

A fibered symplectomorphism is vertical if $(q;p) = \big(q; p + dg(q)\big)$ for a C^2 function $g : \mathbb{R}^{d+1} \to \mathbb{R}$.

It is well known that any fibered symplectomorphism is a composition of vertical and homogeneous symplectomorphisms.

Remarks about the normal form

The proof of the perturbation theorem was based on a wrong normal form given by Figalli and Rifford [FR15]. In contrary to what was believed in the literature, we proved that by using only fibered homogeneous symplectic change of coordinates it is impossible to derive the assertions of the normal form.

However, for positively homogeneous Hamiltonians we obtained a Fermi type normal form using only homogeneous symplectomorphisms. This normal form implies the normal form obtained by Li and Nirenberg [LN05] for Finsler metrics. In this way we removed the confusion between Li-Nirenberg normal form and a similar desired normal form for non-homogeneous Hamiltonians.

Linearized transition maps

In the coordinate system given by the normal form, consider

$$R^t : \{q_1 = 0\} \cap H^{-1}(0) \to \{q_1 = t\} \cap H^{-1}(0)$$

as a one-parameter family of restricted transition maps along the orbit segment $(te_1, 0)$.

Linearized transition maps

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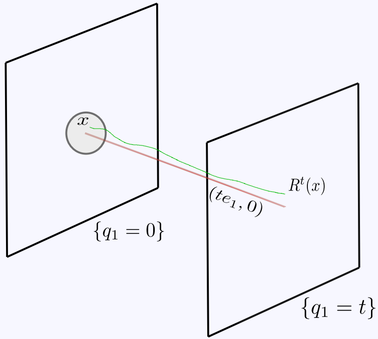
as a one-parameter family of restricted transition maps along the orbit segment $(te_1, 0)$:



Linearized transition maps

Lemma

Assume that a smooth Hamiltonian $H : T^*\mathbb{R}^{d+1} \to \mathbb{R}$ takes $(te_1, 0) \in H^{-1}(0)$ as an orbit segment where $t \in [0, \tau]$ for some $\tau > 0$, and H satisfies the assertions of the normal form on this orbit segment. Then, $L(t) := dR^t(0)$ solves the differential equation

$$L'(t) = Y(t)L(t),$$

where $Y(t) := \mathbb{J}\partial^2_{\hat{x}^2} H(te_1, 0)$, and

$$R^t : \{q_1 = 0\} \cap H^{-1}(0) \to \{q_1 = t\} \cap H^{-1}(0)$$

is the one-parameter family of transition maps associated to the segment $(te_1, 0)$, $t \in [0, \tau]$.

We are using the notation $q = (q_1, \hat{q}) \in \mathbb{R} \times \mathbb{R}^d$, $p = (p_1, \hat{p}) \in \mathbb{R} \times \mathbb{R}^d$, $\hat{x} = (\hat{q}, \hat{p})$ and $x_1 = (q_1, p_1)$.

Linearized transition maps

Corollary

Assume that $H : T^{*}\mathbb{R}^{d+1} \to \mathbb{R}$ *is smooth and for some* $\ > 0$; $\ (t) = (te_1, 0) \in H^{-1}(0)$; *where* $t \in [0, \ ]$; *is an orbit segment of the Hamiltonian vector field of* H. *Moreover, suppose that* H *satisfies the assertions of the normal form on* $\ (t)$ *for all* $t \in [0, \ ]$. *Let*

$$R_u^t : \{q_1 = 0\} \cap (H+u)^{-1}(0) \to \{q_1 = t\} \cap (H+u)^{-1}(0), \quad u \in C^{\infty}(\mathbb{R}^{d+1});$$

be the one-parameter family of transition maps along the orbit segment $(te_1, 0)$ *and the Hamiltonian vector field of* $H + u$. *Then* $L_u(t) := dR_u^t(0)$ *solves the differential equation*

$$L_u'(t) = Y_u(t) L_u(t);$$

where $Y_u(t) := J\partial^2_{x^2}(H + u)(te_1, 0)$.

Linearized transition maps in view point of control theory

The equation $L^0_u(t) = Y_u(t)L_u(t)$ can be viewed as a control problem where $@^2_{\hat{q}^2} u(te_1)$ is the control.

Inspired by Rifford and Ruggiero [RR12], by finding sufficient conditions for local controllability of the above control problem, we managed to prove a perturbation theorem for non-convex Hamiltonians.

Perturbation theorem for non-convex Hamiltonians

Theorem

Suppose that (t) 2 H $^{1}(k)$ is a regular periodic orbit of a smooth Hamiltonian $H : T$ R^{d+1} ! R: For potentials u 2 $C^{1}(\mathrm{R}^{d+1})$; where $C^{1}(\mathrm{R}^{d+1})$: Consider

$$P_u(\; ; \;) : \quad \backslash \; (H+u) \quad ^{1}(k) \; ! \quad \backslash \; (H+u) \quad ^{1}(k)$$

as the restricted Poincaré map with respect to (t) and Hamiltonian vector field of $H + u$; where is a transverse section to (t): Assume that (t) admits a neat time t_0 such that (t_0) 2 $_H$: Then, the map $F(\; ; H + u)$ defined as

$$C^{1}(M) \; 3 \; u \; 7! \; dP_u \; 2 \; Sp(2d)$$

is weakly open.

We proved that the assumption " (t) admits a neat time t_0 such that (t_0) 2 $_H$:" is a necessary assumption for the above theorem.

Genericity results

Using the perturbation theorem among other methods we prove the following theorem.

Theorem (1)

Assume that $H : T \quad M \to \mathbb{R}$ is a smooth Hamiltonian defined on the cotangent bundle of a closed smooth manifold M: Let $\quad Sp(2d)$ be a given $F \quad$ nowhere dense subset invariant under congugacy. For a given $k \in \mathbb{R}$; there exists a $G \quad$ dense subset $G \quad C^{1}(M)$ such that for all $u \in G$ the k-energy level of $H + u$ is regular; Moreover, if $(t) \in (H + u) \quad ^{1}(k)$ is a periodic orbit that admits a neat time t_0 such that $(t_0) \in \quad _H$; then the linearized restricted Poincaré map associated to $\quad$ and Hamiltonian vector field of $H + u$ does not belong to $\quad$:

Genericity results

Theorem (2)

Let $H : T^*M \to \mathbb{R}$ be a smooth Hamiltonian defined on the cotangent bundle of a smooth manifold M. Assume that $\Sigma_H \subset T^*M$ satisfies Hypothesis 1. There exists a G_δ dense subset $\mathcal{G} \subset C^\infty(M)$ such that for all $u \in \mathcal{G}$, the Hamiltonian vector field associated to $H + u$ has the following property:
For each orbit $\gamma(t)$ of $H + u$ and each time t_0 such that $\partial_p H(\gamma(t_0)) \neq 0$, there exist an open neighborhood $I \subset \mathbb{R}$ around t_0 so that
$\gamma(I \setminus t_0) \cap \Sigma_H = \emptyset.$

Consider $\hat{H}^\infty(T^*M)$ as the space of all smooth Hamiltonians defined on T^*M that are satisfying Hypothesis 1.
For a given Hamiltonian $H \in \hat{H}^\infty(T^*M)$, the set of potentials

$$\{u \in C^\infty(M) \mid \text{Hamiltonian vector field associated to } H + u \text{ satisfies } (g)\}$$

is a G_δ dense subset of $C^\infty(M)$, where property (g) is described below

(g) all orbits that are admitting a neat time are also admitting a neat time in the complement of $\Sigma_H \subset T^*M$.

Genericity results

PhD Thesis Defense

Shahriar Aslani

From Theorem (1) and Theorem (2) we conclude the following theorem which implies the bumpy metric theorem.

Theorem (3)

Assume that $H : T\ \ M\ \ !\ \ \mathrm{R}$ is a smooth Hamiltonian defined on the cotangent bundle of a smooth manifold M: Suppose that $\ _H$ satisfies Hypothesis 1. Let $\quad Sp(2d)$ be a $F\ $ nowhere dense subset invariant under congugacy. For a given $k\ 2\ \mathrm{R}$; there exists a $G\ $ dense subset $G \quad C^1(M)$ such that for all $u\ 2\ \mathrm{G}$ the Hamiltonian $H + u$ has the following property:
$(H + u)\ \ ^1(k)$ is a regular energy level. Furthermore, if $\ (t)\ 2\ (H + u)\ \ ^1(k)$ is a periodic orbit that admits a neat time, then the linearized restricted Poincaré map associated to $\ (t)$ and Hamiltonian vector field of $H + u$ does not belong to $\ $:

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